

## Threshold resummation beyond leading eikonal level

---

**Georges Grunberg\***

*Centre de Physique Théorique, École Polytechnique,  
91128 Palaiseau Cedex, France*

*E-mail: georges.grunberg@cpht.polytechnique.fr*

The modified evolution equation for parton distributions of Dokshitzer, Marchesini and Salam is extended to non-singlet Deep Inelastic Scattering coefficient functions and the physical evolution kernels which govern their scaling violation. Considering the  $x \rightarrow 1$  limit, it is found that the leading next-to-eikonal logarithmic contributions to the momentum space physical kernels at any loop order can be expressed in term of the one loop cusp anomalous dimension, a result which can presumably be extended to all orders in  $(1-x)$ . Similar results hold for fragmentation functions in semi-inclusive  $e^+e^-$  annihilation. The method does not work for subleading next-to-eikonal logarithms, but, in the special case of the  $F_1$  and  $F_T$  structure and fragmentation functions, there are hints of the possible existence of an underlying Gribov-Lipatov like relation.

*XVIII International Workshop on Deep-Inelastic Scattering and Related Subjects  
April 19 -23, 2010  
Convitto della Calza, Firenze, Italy*

---

\*Speaker.

## 1. Threshold resummation in physical evolution kernels

Consider a generic deep inelastic scattering (DIS) *non-singlet* structure function  $\mathcal{F}(x, Q^2) = \{2F_1(x, Q^2), F_2(x, Q^2)/x\}$  at large  $Q^2 \gg \Lambda^2$ . We shall be interested in the elastic limit  $x \rightarrow 1$  where the final state mass  $W^2 \sim (1-x)Q^2 \ll Q^2$ . In this limit, large threshold  $\ln(1-x)$  logarithms appear. Their resummation is by now standard [1, 2], but usually performed in moment space. However, the result can also be expressed *analytically in momentum space* at the level of so-called “physical evolution kernels” which account for the *physical* scaling violation:

$$\frac{\partial \mathcal{F}(x, Q^2)}{\partial \ln Q^2} = \int_x^1 \frac{dz}{z} K(z, a_s(Q^2)) \mathcal{F}(x/z, Q^2) \equiv K \otimes \mathcal{F} , \quad (1.1)$$

where the “physical evolution kernel”  $K(x, a_s)$  ( $a_s = \alpha_s/4\pi$  is the  $\overline{MS}$  coupling) embodies *all* the *perturbative* information about  $\mathcal{F}$ . For  $x \rightarrow 1$  threshold resummation yields [3]:

$$K(x, a_s(Q^2)) \sim \left[ \frac{\mathcal{J}((1-x)Q^2)}{1-x} \right]_+ + B_\delta^{DIS}(a_s(Q^2)) \delta(1-x) , \quad (1.2)$$

where  $\mathcal{J}(Q^2)$  is a “physical anomalous dimension” (a renormalization scheme invariant quantity), related to the standard “cusp”  $A(a_s) = \sum_{i=1}^\infty A_i a_s^i$  and final state “jet function”  $B(a_s) = \sum_{i=1}^\infty B_i a_s^i$  anomalous dimensions by:

$$\mathcal{J}(Q^2) = A(a_s(Q^2)) + \beta(a_s(Q^2)) \frac{dB(a_s(Q^2))}{da_s} \equiv \sum_{i=1}^\infty j_i a_s^i(Q^2) . \quad (1.3)$$

The renormalization group invariance of  $\mathcal{J}(Q^2)$  yields the standard relation:

$$\begin{aligned} \mathcal{J}((1-x)Q^2) &= j_1 a_s + a_s^2 [-j_1 \beta_0 L_x + j_2] \\ &\quad + a_s^3 [j_1 \beta_0^2 L_x^2 - (j_1 \beta_1 + 2j_2 \beta_0) L_x + j_3] + \mathcal{O}(a_s^4) , \end{aligned} \quad (1.4)$$

where  $L_x \equiv \ln(1-x)$  and  $a_s = a_s(Q^2)$ , from which the structure of *all* the eikonal logarithms in  $K(x, a_s(Q^2))$ , which can be absorbed into the *single* scale  $(1-x)Q^2$ , can thus be derived.

However, no analogous result holds at the next-to-eikonal level (except [4] at large- $\beta_0$ ). Indeed, expanding

$$K(x, a_s) = K_0(x) a_s + K_1(x) a_s^2 + K_2(x) a_s^3 + \mathcal{O}(a_s^4) , \quad (1.5)$$

the  $K_i$ ’s can be determined as combinations of splitting and coefficient functions. One gets:

$$K_0(x) = P_0(x) = k_{10} p_{qq}(x) + \Delta_1 \delta(1-x) , \quad (1.6)$$

with  $k_{10} = A_1$  and  $p_{qq}(x) = \frac{x}{1-x} + \frac{1}{2}(1-x)$ . Moreover for  $x \rightarrow 1$  one finds [5, 6], barring delta function contributions:

$$\begin{aligned} K_1(x) &= \frac{x}{1-x} (k_{21} L_x + k_{20}) + (h_{21} L_x + h_{20}) + \mathcal{O}((1-x) L_x) \\ K_2(x) &= \frac{x}{1-x} (k_{32} L_x^2 + k_{31} L_x + k_{30}) + (h_{32} L_x^2 + h_{31} L_x + h_{30}) + \mathcal{O}((1-x) L_x^2) . \end{aligned} \quad (1.7)$$

Despite the similar logarithmic structure, the *next-to-eikonal* logarithms  $h_{ij}$  cannot [5] be obtained from a standard renormalization group resummation analogous to the one used (eq.(1.4)) for the *eikonal* logarithms  $k_{ij}$ .

## 2. An alternative approach: the modified physical kernel

Instead, consider [7] a modified physical evolution equation, similar to the one used in [8] (see also [9]) for parton distributions:

$$\frac{\partial \mathcal{F}(x, Q^2)}{\partial \ln Q^2} = \int_x^1 \frac{dz}{z} K(z, a_s(Q^2), \lambda) \mathcal{F}(x/z, Q^2/z^\lambda), \quad (2.1)$$

where the arbitrary parameter  $\lambda$  shall be set to 1 at the end. Expanding  $\mathcal{F}(y, Q^2/z^\lambda)$  around  $z = 1$ , one can relate  $K(x, a_s, \lambda)$  to  $K(x, a_s)$ :

$$K(x, a_s, \lambda) = K(x, a_s) + \lambda [\ln x K(x, a_s, \lambda)] \otimes K(x, a_s) + \dots \quad (2.2)$$

Solving perturbatively, one finds that for  $x \rightarrow 1$  the corresponding expansion coefficients  $K_i(x, \lambda)$  satisfy the analogue of eq.(1.7), with the *same* coefficients  $k_{ji}$ 's of the eikonal logarithms, but with the coefficients of the *leading* next-to-eikonal logarithms given by:

$$\begin{aligned} h_{21}(\lambda) &= h_{21} - \lambda k_{10}^2 \\ h_{32}(\lambda) &= h_{32} - \lambda \frac{3}{2} k_{21} k_{10}. \end{aligned} \quad (2.3)$$

Setting now  $\lambda = 1$ , one observes that both  $h_{21}(\lambda = 1)$  and  $h_{32}(\lambda = 1)$  *vanish*, which means that  $h_{21} = k_{10}^2 = A_1^2 = 16C_F^2$  and  $h_{32} = \frac{3}{2} k_{21} k_{10} = -\frac{3}{2} \beta_0 A_1^2 = -24\beta_0 C_F^2$ , which agree with the exact results in [5, 6]. It should be stressed that, whereas  $h_{21}$  is contributed by the two loop splitting function alone (and thus one simply recovers in this case the result of [8]),  $h_{32}$  is instead contributed *only* by the one and two loop coefficient functions, which represents a new result. Similar results are obtained for the coefficients  $h_{ji}$  ( $j = i + 1$ ) of the *leading* next-to-eikonal logarithms at any loop order, which can all be expressed in term of the one loop cusp anomalous dimension *assuming* the corresponding  $h_{ji}(\lambda)$  vanish for  $\lambda = 1$ . In particular, one predicts  $h_{43} = \frac{4}{3} k_{10} k_{32} + \frac{1}{2} k_{21}^2 = \frac{11}{6} \beta_0^2 A_1^2 = \frac{88}{3} \beta_0^2 C_F^2$ , which is correct [5, 6], and  $h_{54} = \frac{5}{4} k_{10} k_{43} + \frac{5}{6} k_{21} k_{32} = -\frac{25}{12} \beta_0^3 A_1^2 = -\frac{100}{3} \beta_0^3 C_F^2$ , which remains to be checked.

Similar results are obtained for the coefficients  $f_{ji}$  ( $j = i + 1$ ) of the *leading* next-to-next-to eikonal logarithms, defined by:

$$K_i(x)|_{\text{LL}} = L_x^i [p_{qq}(x) k_{ji} + h_{ji} + (1-x)f_{ji} + \mathcal{O}((1-x)^2)], \quad (2.4)$$

where the *full* one loop prefactor  $p_{qq}(x)$  should be used in the leading term to define the  $f_{ji}$ 's. The corresponding  $f_{ji}(\lambda)$  coefficients in  $K_i(x, \lambda)$  are given by:

$$f_{21}(\lambda) = f_{21} + \lambda \frac{1}{2} k_{10}^2 \quad (2.5)$$

$$\begin{aligned}
f_{32}(\lambda) &= f_{32} - \lambda \left( -\frac{3}{4} k_{10} k_{21} + k_{10} h_{21} \right) + \lambda^2 \frac{1}{2} k_{10}^3 \\
f_{43}(\lambda) &= f_{43} - \lambda \left( -\frac{2}{3} k_{10} k_{32} + \frac{1}{2} (h_{21} - \frac{1}{2} k_{21}) k_{21} + k_{10} h_{32} \right) + \lambda^2 k_{10}^2 k_{21} ,
\end{aligned}$$

where one notes the presence of contributions *quadratic* in  $\lambda$ . Assuming the  $f_{ji}(\lambda)$ 's vanish for  $\lambda = 1$ , the resulting predictions for the  $f_{ji}$ 's ( $j = i + 1$ ) are again found to agree with the exact results of [6].

### 3. Fragmentation functions

Similar results hold for physical evolution kernels associated to fragmentation functions in semi-inclusive  $e^+e^-$  annihilation (SIA), provided one sets  $\lambda = -1$  in the modified evolution equation:

$$\frac{\partial \mathcal{F}_{SIA}(x, Q^2)}{\partial \ln Q^2} = \int_x^1 \frac{dz}{z} K_{SIA}(z, a_s(Q^2), \lambda) \mathcal{F}_{SIA}(x/z, Q^2/z^\lambda) , \quad (3.1)$$

where  $\mathcal{F}_{SIA} = \{\mathcal{F}_T, \mathcal{F}_{T+L}\}$  denotes a generic *non-singlet* fragmentation function (I use the notation of [6]). At the *leading* eikonal level, threshold resummation [10] can be summarized in the standard SIA physical evolution kernel by:

$$K_{SIA}(x, a_s(Q^2)) \sim \left[ \frac{\mathcal{J}((1-x)Q^2)}{1-x} \right]_+ + B_\delta^{SIA}(a_s(Q^2)) \delta(1-x) , \quad (3.2)$$

where the ‘‘physical anomalous dimension’’  $\mathcal{J}(Q^2)$  (hence the  $k_{ji}$ 's) are the *same* for DIS and SIA, as follows from the results in [11]. Assuming the *leading* threshold logarithms *vanish* beyond the leading eikonal level in the *modified* SIA evolution kernel for  $\lambda = -1$ , and setting  $\lambda = -1$  in eq.(2.3) and (2.5), one derives predictions for  $h_{ji}^{SIA}$  and  $f_{ji}^{SIA}$  ( $j = i + 1$ ) which again agree with the exact results of [6]. In particular, one finds that  $h_{ji}^{SIA} = -h_{ji}$ .

### 4. Subleading next-to-eikonal logarithms

The previous approach *does not* work for *subleading* next-to-eikonal logarithms, namely the latter do not vanish in the modified physical evolution kernels for  $\lambda = \pm 1$ . The following facts are nevertheless worth quoting:

- At large  $\beta_0$ , we have a generalization [4] of the leading eikonal *single scale* ansatz (which takes care of *all* subleading logarithms) to *any* eikonal order:

$$\begin{aligned}
K(x, Q^2)|_{\text{large } \beta_0} &= \left[ \frac{x}{1-x} \mathcal{J}(W^2)|_{\text{large } \beta_0} \right]_+ + (\delta(1-x) \text{term}) \\
&\quad + \mathcal{J}_0(W^2)|_{\text{large } \beta_0} + (1-x) \mathcal{J}_1(W^2)|_{\text{large } \beta_0} + \dots
\end{aligned} \quad (4.1)$$

where  $W^2 = (1-x)Q^2$ , and the  $\mathcal{J}_i$ 's (*except* the leading eikonal one) are structure function dependent. A similar result holds for  $K_{SIA}(x, Q^2)|_{\text{large } \beta_0}$ .

- There are remarkable relations between the *momentum space next-to-leading* threshold logarithms of the (DIS)  $F_1$  and the corresponding (SIA)  $F_T$  transverse fragmentation function physical evolution kernels at the *next-to-eikonal* level. Namely, using the moment space results of [6], one can derive the following *momentum space* relations:

1) At two loop for the  $\mathcal{O}(L_x^0)$  next-to-eikonal constant term:

$$\begin{aligned} h_{20}^{(F_1)} &= h_{20}^{(F_1)} \Big|_{\text{large } \beta_0} + \Delta h_{20} \\ h_{20}^{(F_T)} &= h_{20}^{(F_T)} \Big|_{\text{large } \beta_0} - \Delta h_{20} , \end{aligned} \quad (4.2)$$

with  $h_{20}^{(F_1)} \Big|_{\text{large } \beta_0} = -11\beta_0 C_F$ ,  $h_{20}^{(F_T)} \Big|_{\text{large } \beta_0} = 7\beta_0 C_F$ , and  $\Delta h_{20} = A_1 \Delta_1 = 12C_F^2$ .

2) At three loop for the *single*  $\mathcal{O}(L_x)$  next-to-eikonal logarithms:

$$\begin{aligned} h_{31}^{(F_1)} &= h_{31}^{(F_1)} \Big|_{\text{large } \beta_0} + \Delta h_{31} \\ h_{31}^{(F_T)} &= h_{31}^{(F_T)} \Big|_{\text{large } \beta_0} - \Delta h_{31} , \end{aligned} \quad (4.3)$$

with  $h_{31}^{(F_1)} \Big|_{\text{large } \beta_0} = -2\beta_0 h_{20}^{(F_1)} \Big|_{\text{large } \beta_0} = 22C_F\beta_0^2$ ,  $h_{31}^{(F_T)} \Big|_{\text{large } \beta_0} = -2\beta_0 h_{20}^{(F_T)} \Big|_{\text{large } \beta_0} = -14C_F\beta_0^2$ , and:

$$\Delta h_{31} = 2A_1 A_2 - 20\beta_0 C_F C_A + 20\beta_0 C_F^2 . \quad (4.4)$$

3) At four loop for the *double*  $\mathcal{O}(L_x^2)$  next-to-eikonal logarithms:

$$\begin{aligned} h_{42}^{(F_1)} &= h_{42}^{(F_1)} \Big|_{\text{large } \beta_0} + \Delta h_{42} \\ h_{42}^{(F_T)} &= h_{42}^{(F_T)} \Big|_{\text{large } \beta_0} - \Delta h_{42} , \end{aligned} \quad (4.5)$$

with  $h_{42}^{(F_1)} \Big|_{\text{large } \beta_0} = 3\beta_0^2 h_{20}^{(F_1)} \Big|_{\text{large } \beta_0} = -33C_F\beta_0^3$ ,  $h_{42}^{(F_T)} \Big|_{\text{large } \beta_0} = 3\beta_0^2 h_{20}^{(F_T)} \Big|_{\text{large } \beta_0} = 21C_F\beta_0^3$ , and:

$$\Delta h_{42} = -24\beta_1 C_F^2 + 45\beta_0^2 C_F C_A - 178\beta_0^2 C_F^2 - (47 - 10\zeta_2)\beta_0 C_F C_A - (60 - 140\zeta_2)\beta_0 C_F^2 C_A - 16\beta_0 C_F^3 . \quad (4.6)$$

The large- $\beta_0$  parts are consistent with eq.(4.1), while the remaining  $\pm\Delta h_{ij}$  corrections are suggestive of an underlying (yet to be discovered) Gribov-Lipatov like relation [14].

- *No such relations* exist between the DIS  $F_2$  structure function and the corresponding total angle-integrated  $F_{T+L}$  fragmentation function. This fact suggests to focus instead on the *momentum space* physical evolution kernels of the *longitudinal* structure [12, 13] and fragmentation functions. Indeed, some observations in [6] do suggest that the  $\mathcal{O}(1/(1-x))$  part of the spacelike and timelike longitudinal evolution kernels might actually be *identical* to any logarithmic accuracy.

## 5. Conclusions

- Using a kinematically modified [8] physical evolution equation, evidence has been given that the *leading* threshold logarithms at *any* eikonal order in the *momentum space* DIS and SIA *non-singlet* physical evolution kernels can be expressed in term of the *one loop* cusp anomalous dimension  $A_1$ , which represents the *first step* towards threshold resummation *beyond* the leading eikonal level. This result also explains the observed *universality* [5, 6] of the *leading* logarithmic contributions to the physical kernels of the various non-singlet structure functions at *any* order [6] in  $1-x$ .
- The present approach *does not* work for *subleading* next-to-eikonal logarithms. However, there are hints of the possible existence of an underlying (yet to be understood) Gribov-Lipatov like relation in the special case of the  $F_1$  DIS structure function and the corresponding  $F_T$  SIA transverse fragmentation function.

## References

- [1] G. Sterman, *Nucl. Phys.* **B281** (1987) 310.
- [2] S. Catani and L. Trentadue, *Nucl. Phys.* **B327** (1989) 323.
- [3] E. Gardi and G. Grunberg, *Nucl. Phys.* **B 794** (2008) 61 [arXiv:0709.2877 [hep-ph]].
- [4] G. Grunberg, arXiv:0710.5693 [hep-ph].
- [5] G. Grunberg and V. Ravindran, *JHEP* **0910** (2009) 055 [arXiv:0902.2702 [hep-ph]].
- [6] S. Moch and A. Vogt, *JHEP* **0911** (2009) 099 [arXiv:0909.2124 [hep-ph]].
- [7] G. Grunberg, *Phys. Lett. B* **687** (2010) 405 [arXiv:0911.4471 [hep-ph]].
- [8] Yu. L. Dokshitzer, G. Marchesini and G. P. Salam, *Phys. Lett. B* **634** (2006) 504 [arXiv:hep-ph/0511302].
- [9] B. Basso and G. P. Korchemsky, *Nucl. Phys.* **B 775** (2007) 1 [arXiv:hep-th/0612247].
- [10] M. Cacciari and S. Catani, *Nucl. Phys.* **B 617** (2001) 253 [arXiv:hep-ph/0107138].
- [11] S. Moch and A. Vogt, *Phys. Lett. B* **680** (2009) 239 [arXiv:0908.2746 [hep-ph]].
- [12] S. Moch and A. Vogt, *JHEP* **0904** (2009) 081 [arXiv:0902.2342 [hep-ph]].
- [13] G. Grunberg, arXiv:0910.3894 [hep-ph].
- [14] V. N. Gribov and L. N. Lipatov, *Sov. J. Nucl. Phys.* **15** (1972) 438, *ibid.* **15** (1972) 675.